Suggested Solutions HW4, Class 151A, Spring 2012

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2.2.7

Since $g'(x) = \frac{1}{4}\cos\frac{x}{2}$, g is continuous and g' exists on $[0, 2\pi]$. Further, $g'(x) = 0$ only when $x = \pi$, so that $g(0) = g(2\pi) = \pi \leq g(x) =\leq g(\pi) = \pi + \frac{1}{2}$ and $|g'(x)| \leq \frac{1}{4}$, for $0 \leq x \leq 2\pi$. Theorem 2.3 implies that a unique fixed point p exists in $[0, 2\pi]$. With $k = \frac{1}{4}$ and $p_0 = \pi$, we have $p_1 = \pi + \frac{1}{2}$. Corollary 2.5 implies that

$$|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0| = \frac{2}{3}\left(\frac{1}{4}\right)^n.$$

For the bound to be less than 0.1, we need $n \geq 4$. However, $p_3 = 3.626996$ is accurate to within 0.01.

2.2.19(a)

Let $g(x) = \frac{x}{2} + \frac{1}{x}$ and consider the interval $I = \left[\sqrt{2}, x_0\right]$. We'd like to apply Theorem 2.4 to this function.

First, note that for all $x \in I$,

$$\begin{aligned} 0 &\leq |g'(x)| \\ &= \left|\frac{1}{2} - \frac{1}{x^2}\right| \\ &= \frac{1}{2} - \frac{1}{x^2} \\ &\leq \frac{1}{2} < 1 \end{aligned}$$

This tells us that we can set $k = 1/2$. We also need to check that $g(I) \subset I$. To do this, we look for the minimum and maximum of g on I. First, note that $x = \sqrt{2}$ is the only value in the domain for which $g'(x) = 0$ implying the max and min of g on I occur at the endpoints. We have,

$$\begin{aligned} g(\sqrt{2}) &= \sqrt{2} \\ g(x_0) &= \frac{x_0}{2} + \frac{1}{x_0} \end{aligned}$$

Now, since $g'(x) > 0$, for all $x \in I$, we have $g(x_0) > g(\sqrt{2}) = \sqrt{2}$. But, is $g(x_0) \leq x_0$? We have,

$$
\begin{aligned}
x_0 &\geq \sqrt{2} \Rightarrow \\
x_0^2 &\geq 2 \Rightarrow \\
2x_0^2 &\geq x_0^2 + 2 \Rightarrow \\
x_0 &\geq \frac{x_0^2 + 2}{2x_0} \\
&= \frac{x_0}{2} + \frac{1}{x_0} \\
&= g(x_0)
\end{aligned}
$$

Thus $g(x_0) \in I$. In summary, we've show that $g(I) \subset I$ and that there exists $k \in (0,1)$ such that $|g'(x)| < k$ for all $x \in I$. This implies g has a unique fixed point in I and we've already identified it: $g(\sqrt{2}) = \sqrt{2}$. Finally, note that the given sequence is simply the one defined by the Theorem, so, the sequence converges to the fixed point, $\sqrt{2}$.

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Alternative solution **without using Thm. 2.4.**:
Let $g(x) = \frac{x}{2} + \frac{1}{x}$. For $x \neq 0$, we get $g'(x) = \frac{1}{2} - \frac{1}{x^2}$. If $x > \sqrt{2}$, then $\frac{1}{x^2} < \frac{1}{2}$, hence $g'(x) > 0$. Also, $g(\sqrt{2}) = \sqrt{2}$. Suppose that, $x_0 > \sqrt{2}$. Then,

$$x_1 - \sqrt{2} = g(x_0) - g(\sqrt{2}) = g'(\xi)(x_0 - \sqrt{2}), \qquad \text{(0-th Taylor exp.)}$$

where $\sqrt{2} < \xi < x_1$. Thus, $x_1 - \sqrt{2} > 0$ and $x_1 > \sqrt{2}$. This implies

$$
\begin{aligned}
& x_1 = \frac{x_0}{2} + \frac{1}{x_0} < \frac{x_0}{2} + \frac{1}{\sqrt{2}} = \frac{x_0 + \sqrt{2}}{2} \\
\Rightarrow \quad & 2x_1 < x_0 + \sqrt{2} \quad \Rightarrow \quad 2x_1 < x_0 + x_0 \quad \Rightarrow \quad \sqrt{2} < x_1 < x_0.
\end{aligned}
$$

By induction we have

$$\sqrt{2} < x_{n+1} < x_n < \cdots < x_1 < x_0.$$

Thus, $\{x_n\}$ is a decreasing sequence with the lower bound $\sqrt{2}$, hence it must converge. The limiting value is given by

$$p = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \left(\frac{x_{n-1}}{2} + \frac{1}{x_{n-1}} \right) = \frac{p}{2} + \frac{1}{p} \quad \Rightarrow \quad p = \pm\sqrt{2}.$$

Since $x_n > \sqrt{2}$ for all n, we have $\lim_{n\to\infty} x_n = \sqrt{2}$.

2.3.4

Let $f(x) = -x^3 - cos(x)$. With initial values $p_0 = -1$ and $p_1 = 0$, find p_3.
(a) Use the **secant method**:
We have

$$f(p_0) = f(-1) = 1 - \cos(-1) \quad \text{and} \quad f(p_1) = f(0) = -1,$$

thus

$$p_2 = p_1 - \frac{f(p_1)(p_1-p_0)}{f(p_1)-f(p_0)} = \frac{1}{\cos(-1)-2} \approx -0.685073$$

$$p_3 = p_2 - \frac{f(p_2)(p_2-p_1)}{f(p_2)-f(p_1)} = \frac{1}{\cos(-1)-2} - \frac{\left(\frac{-1}{(\cos(-1)-2)^3} - \cos\left(\frac{1}{\cos(-1)-2}\right)\right)\left(\frac{1}{\cos(-1)-2}\right)}{\frac{-1}{(\cos(-1)-2)^3} - \cos\left(\frac{-1}{(\cos(-1)-2)^3}\right) + 1}$$

$$\approx -1.25208$$

(b) Use the **method of false position**:
From above in (a) we can observe $f(p_2)f(p_1) > 0$. Thus, use p_2 and p_0 in the secant method to correct the false position, i.e. we choose p_3 as the intercept of the line joining $(p_0, f(p_0))$ and $(p_2, f(p_2))$ and then interchange the indices on p_0 and p_1. We obtain:

$$p_3 = p_2 - \frac{f(p_2)(p_2-p_0)}{f(p_2)-f(p_0)} = \frac{1}{\cos(-1)-2} - \frac{\left(\frac{-1}{(\cos(-1)-2)^3} - \cos\left(\frac{1}{\cos(-1)-2}\right)\right)\left(\frac{1}{\cos(-1)-2}+1\right)}{\frac{-1}{(\cos(-1)-2)^3} - \cos\left(\frac{-1}{(\cos(-1)-2)^3}\right) - (1-\cos(-1))}$$

$$\approx -0.841355$$

2.3.5(a)

$$f(x) = x^3 - 2x^2 - 5 = 0, \quad \text{interval: } [1,4], \quad f'(x) = 3x^2 - 4x$$

An analytical solution within the interval is given by: $p \approx 2.69065$.
Select $p_0 = 2$, and apply Newton's method: $p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$

$$p_1 = 2 - \frac{2^3 - 2\cdot 2^2 - 5}{3\cdot 2^2 - 4\cdot 2} = 2 + \frac{5}{4} = 3.25$$
$$p_2 = 3.25 - \frac{3.25^3 - 2\cdot 3.25^2 - 5}{3\cdot 3.25^2 - 4\cdot 3.25} = 2.811036789$$
$$p_3 = p_2 - \frac{p_2^3 - 2\cdot p_2^2 - 5}{3\cdot p_2^2 - 4\cdot p_2} = 2.697989502$$
$$p_4 = 2.690677153$$

Hence the absolute error $|p_4 - p| < 10^{-4}$ reaches the desired tolerance after 4 iterations.

2.3.14

Define distance $[d(x)]^2 = (x-2)^2 + (\frac{1}{x} - 1)^2 = x^2 - 4x + 4 + \frac{1}{x^2} - \frac{2}{x} + 1 =: g(x)$.
We are interested in finding the minimum of $g(x)$. Hence, we look at the necessary optimality condition, i.e. we will compute a real positive root of the function

$$f(x) := g'(x) = 2x - 4 - \frac{2}{x^3} + \frac{2}{x^2}.$$

Thus, $f'(x) = 2 + \frac{6}{x^4} - \frac{4}{x^3}$.
Select $p_0 = 2$, and apply Newton's method: $p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$

$$\begin{aligned}
p_1 &= 2 - \frac{2 \cdot 2 - 4 - \frac{2}{2^3} + \frac{2}{2^2}}{2 + \frac{6}{2^4} - \frac{4}{2^3}} = 1.8666667 \\
p_2 &= p_1 - \frac{f(p_1)}{f'(p_1)} = 1.8667604 \\
p_3 &= 1.8667604
\end{aligned}$$

Since the absolute difference of the iterates $|p_3 - p_2| < 10^{-4}$ reaches the desired tolerance, we can stop after 3 iterations and obtain the point $(1.8667604, 0.5356874)$.

2.3.33

The shut out probability, P, should be at least 0.5. Hence,

$$f(p) := P - 0.5 = \frac{1+p}{2}\left(\frac{p}{1-p+p^2}\right)^{21} - \frac{1}{2} \geq 0.$$

We have to study the case $f(p) = 0$ and use the secant method here,

$$p_{n+1} = p_n - \frac{f(p_n)(p_n - p_{n-1})}{f(p_n) - f(p_{n-1})}, \quad n \geq 1.$$

We select $p_0 = 0.5$ and $p_1 = 0.9$, then

$$\begin{aligned}
p_2 &= 0.9 - \frac{f(0.9)(0.9 - 0.5)}{f(0.9) - f(0.5)} = 0.765484401 \\
p_3 &= 0.837792785 \\
p_4 &= 0.843060708 \\
p_5 &= 0.842301911 \\
p_6 &= 0.842304789
\end{aligned}$$

Since the absolute difference of the iterates $|p_6 - p_5| < 10^{-3}$ reaches the desired tolerance, we can stop after 6 iterations, and get $p \geq p_6$ as the resulting probability for A winning any specific rally, such that $P \geq 0.5$.

2.4.6(a)

$$\text{Given sequence:} \qquad p_n = \frac{1}{n}, \quad n \geq 1.$$

Note that $\lim_{n\to\infty} p_n = 0 =: p$. Since

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n\to\infty} \frac{|p_{n+1}|}{|p_n|} = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1 + \frac{1}{n}} = 1$$

holds, the convergence is linear. To obtain an absolute error of $5 \cdot 10^{-2}$,

$$|p_n - p| \leq 5 \cdot 10^{-2} \Rightarrow \frac{1}{n} \leq 5 \cdot 10^{-2} \Rightarrow n \geq 20,$$

we need at least 20 steps.